

# One-loop $\beta$ -function of noncommutative scalar $QED_4$

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## Abstract

In this paper we consider the  $\beta$ -function at one-loop approximation for noncommutative scalar QED. The renormalization of the full theory, including the basic vertices, and the renormalization group equation are fully established. Next, the complete set of the one-loop diagrams corresponding to the first-order radiative corrections to the basic functions is considered: gauge, charged scalar and ghost fields self-energies, three- and four-point vertex functions  $\langle \phi^\dagger \phi A \rangle$  and  $\langle \phi^\dagger \phi AA \rangle$ , respectively. We pay special attention to the noncommutative contributions to the renormalization constants. To conclude, the one-loop  $\beta$ -function of noncommutative scalar QED is then computed and comparison to known results is presented.

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## 1 Introduction

Perturbative gauge theories are the most successful models for description of the fundamental interactions in nature, where the coupling constant is small. One of the most successful of these gauge theories is the quantum electrodynamics (QED), a theory that describes the interaction of fermionic charged particles with electromagnetic field. In the QED framework, the one-loop analysis for the physical quantities, such as the electron anomalous magnetic moment and the Lamb shift effect in the energy levels of hydrogen atom, gives us theoretical predictions which are in excellent agreement with experimental data [1]. In fact, the most accurate low-energy measurement of the electromagnetic fine structure constant (the strength of the electromagnetic interaction) originates from the electron anomalous magnetic moment, which was measured precisely using a single electron caught in a Penning trap [2].

As we know, some physical quantities in a field theory do vary with the related energy scale  $\mu$  in which it is considered; for example, the value of the fine structure constant runs with growing energy scale  $\mu$ . In order to account such behavior of all physical quantities, the renormalization group program can suitably be used so that we can define consistently the physical outcomes for a given field theory [3]. In particular, in this context, two functions play a major role, they are the beta and gamma functions, that measure the running of a coupling constant on the energy scale  $\mu$  and the anomalous dimension of correlators, respectively. For instance, perturbative analysis for QED shows that the one-loop  $\beta$ -function is found as  $\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2}$  (with  $N_F = 1$ ) [3]. This physically means that the coupling increases with increasing energy scale so that QED becomes strongly coupled at high energies.

Besides, to describe the dynamics of spinless charged fields interacting with photons, a

suitable framework is the scalar quantum electrodynamics (SQED), and it is known that its one-loop beta function is expressed by  $\beta(e)_{\text{SQED}} = \frac{e^3}{48\pi^2}$  (with  $N_B = 1$ ) [4]. By a simple comparison of this result to that one from QED, it is easily realized that the sign of the both  $\beta$ -functions is positive, showing thus that QED and SQED are infrared free, and moreover that the following identity holds

$$\beta_{\text{QED}} = 4\beta_{\text{SQED}}, \quad (1.1)$$

in which we understand that the coefficient 4 comes from the trace over the gamma matrices due to fermionic loops. We can then observe that the *spin*, as an additional degree of freedom for interacting charged fields in QED, only changes the *intensity* of the  $\beta$ -function and not its *sign*. Besides, if we consider additional degrees of freedom for a spinning charged fields such as color, e.g. in quark matter, then once again it will be observed that the *sign* of the  $\beta$ -function for the model describing the interaction of quarks with photons, does not change. On the other hand, the structure of the  $\beta$ -function for the interaction of quarks with gluons in quantum chromodynamics (QCD) does basically change in comparison to the latter case, since the internal gauge symmetry now is completely changed [3].

As it is known the study of noncommutative gauge theories along the years have uncovered several interesting physical properties [5–7], so that it would be valuable to investigate whether the relation (1.1) is also satisfied for the noncommutative setup. In other words, the posed questions whether or not adding a new degree of freedom to charged fields in a noncommutative spacetime only changes the intensity of the  $\beta$ -function? Furthermore, it is interesting to understand how does the noncommutativity affect on the structure of the  $\beta$ -functions for these QED and SQED theories. The answer to the second question in the case of noncommutative QED has already been studied [8], which is given by

$$\beta(e)_{\text{NC-QED}} = -\frac{e^3}{16\pi^2} \left( \frac{22}{3} - \frac{4N_F}{3} \right). \quad (1.2)$$

The obtained result is independent of the noncommutativity parameter  $\theta$ , but its structure is completely different from its commutative counterpart. Indeed, the one-loop contributions of the relevant graphs to this  $\beta$ -function arise only from the planar parts, while the non-planar parts are all finite. Although the non-planar part of the respective diagrams does not contribute to the  $\beta$ -function at one-loop order, the noncommutative effects are actually presented by means of the new couplings engendered by the noncommutativity of spacetime coordinates. The structure of (1.2) is similar to those from nonAbelian gauge theories, in particular the  $SU(2)$  gauge theory. Besides, we see that for  $N_F = 0$ , the theory reduces to a pure gauge part with a negative  $\beta$ -function which is asymptotically free.

However, the one-loop  $\beta$ -function of the noncommutative SQED has not yet been computed, the exact form of the one-loop  $\beta$ -function for NC-SQED could be naively obtained through the relation (1.2) and the aforementioned considerations in (1.1). To this end, we notice that the contribution  $\frac{22}{3}$  originates just from the pure gauge sector hence this part should also be present in NC-SQED, while the second term, including the matter sector, similar to the commutative case would have a different coefficient (due to its spinless structure). Consequently, we find

$$\beta(e)_{\text{NC-SQED}} = -\frac{e^3}{16\pi^2} \left( \frac{22}{3} - \frac{N_B}{3} \right). \quad (1.3)$$

One of the main goals in this paper is to correctly establish the above result in a more detailed analysis, by considering the interaction among the spinless matter and gauge fields in the noncommutative SQED. Some general discussions on the renormalization of SQED have been considered before [9–11], but with a different scope than ours. More importantly, renormalization of noncommutative gauge theories must be treated carefully and is still subject of analysis [12–14]. Moreover, it is worth to notice the similarity between the  $\beta$ -functions of the noncommutative Abelian gauge theory and commutative non-Abelian gauge theory presented above. Hence, we will discuss along the paper to what extent this similarity holds, and also about the role played by the spin of matter field in the  $\beta$ -function of these theories.

Therefore, the organization of this paper is as follows. In Sec. 2, after introducing the Lagrangian of the NC-SQED model and its invariance under the BRST Slavnov transformations, we present the relevant Feynman rules used in our one-loop calculations. Then in Sec. 3, we discuss the renormalization procedure for the full Lagrangian density and in particular specify the related renormalization constants for the matter, gauge and the interaction terms. Besides we show these renormalization constants satisfy in the Slavnov-Taylor identities. Next, in Sec. 4, the one-loop analysis for the self-energy of the gauge, scalar and the ghost field is performed that yields us the one-loop renormalization constants for the related fields. As well as, the radiative correction to three and four-point vertices, including two scalars-one photon and two scalars-two photons respectively, is carried out in more details, which gives us two renormalization constants corresponding to two vertices. Based on the obtained results for the renormalization constants, the one-loop  $\beta$ -function and also the anomalous dimensions of the scalar, gauge and ghosts fields are presented and discussed in Sec. 5. Finally, the last section 6 is dedicated to the concluding remarks.

## 2 The model

The gauge-fixed dynamics of noncommutative scalar QED is described by the following Lagrangian density

$$\mathcal{L} = (\mathcal{D}^\mu \phi)^\dagger \star (\mathcal{D}_\mu \phi) - m^2 \phi^\dagger \star \phi - \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \mathcal{L}_{g.f} + \mathcal{L}_{gh}, \quad (2.1)$$

where the covariant derivative is defined as  $\mathcal{D}_\mu \phi = \partial_\mu \phi + ie A_\mu \star \phi$  and the field strength tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie [A_\mu, A_\nu]_\star$ , such that  $[\cdot, \cdot]_\star$  is the Moyal bracket. The gauge fixing term is chosen on the Lorenz condition

$$\mathcal{L}_{g.f} = \frac{\xi}{2} B \star B + B \star \partial_\mu A^\mu, \quad (2.2)$$

and  $B$  is the Nakanishi-Lautrup auxiliary field, while the ghost contribution reads

$$\mathcal{L}_{gh} = \partial_\mu \bar{c} \star D^\mu c, \quad (2.3)$$

where we have defined  $D_\mu \bullet = \partial_\mu \bullet + ie [A_\mu, \bullet]_\star$ . The above Lagrangian is invariant under BRST Slavnov transformations [15]

$$sA_\mu = D_\mu c, \quad s\phi = iec \star \phi, \quad sc = iec \star c, \quad s\bar{c} = -B, \quad sB = 0. \quad (2.4)$$

We will show next that the SQED in the noncommutative setup has the behavior upon the discrete symmetries similar to the noncommutative QED [16], this can be seen from the Lagrangian density (2.1) as follows

(i) *Parity*

The transformation of the pure gauge sector under parity has been carried out in [16], where it was shown that the respective pure gauge terms in NC-QED model are invariant under parity without any change in sign of  $\theta$ . To study the behavior of the matter part under parity, we consider the relevant term given by

$$\mathcal{L}_{matter} = ie \left( \partial^\mu \phi^\dagger \star A_\mu \star \phi - \phi^\dagger \star A_\mu \star \partial^\mu \phi \right) + e^2 \phi^\dagger \star A^\mu \star A_\mu \star \phi. \quad (2.5)$$

Under parity, the complex scalar field transforms as  $\phi_P(x') = e^{i\alpha} \phi(x)$  in which the phase  $\alpha$  is arbitrary for a free field [17]. Besides the gauge field components change as  $A_P^0(x') = A^0(x)$  and  $\mathbf{A}_P(x') = -\mathbf{A}(x)$  so that we easily conclude  $\mathcal{L}_{matter}$  does not change under the parity and thus NC-SQED is parity invariant.

(ii) *Charge conjugation*

Under charge conjugation, we have  $A_\mu^c(x) = -A_\mu(x)$  and  $\phi^c(x) = e^{i\eta} \phi^\dagger(x)$ , where  $\eta$  is an arbitrary phase parameter [17]. Taking into account these changes as well as  $\theta \rightarrow -\theta$ , we observe that the first term of  $\mathcal{L}_{matter}$  goes to the second term and vice versa. Furthermore, it is easily seen that the third term in (2.5) does not change and then by considering the invariance of the gauge part as in [16] we consequently deduce that NC-SQED is invariant under charge conjugation transformation (*with*  $\theta \rightarrow -\theta$ ).

(iii) *Time reversal*

Time reversal operator acts on the gauge and the scalar fields as  $A_T^0(x') = A^0(x)$ ,  $\mathbf{A}_T(x') = -\mathbf{A}(x)$  and  $\phi_T(x') = e^{i\zeta} \phi^\dagger(x)$ , respectively. Similar to the aforementioned discussion in charge conjugation, once again we see that the first and the second term of (2.5) transform to each other and the third term remains unchanged, if we assume that  $\theta \rightarrow -\theta$ . Together with invariance of the gauge part, we realize that NC-SQED is time reversal invariant (*with*  $\theta \rightarrow -\theta$ ).

Hence, from the above discussion on the discrete symmetries of NC-SQED, we reach a result which holds also for NC-QED, and was already found in ref. [16]. Actually, it is understood that NC-SQED is parity invariant without assumption  $\theta \rightarrow -\theta$  and hence this is the same as the commutative setup. While in order to have charge conjugation and time reversal invariance, we should take into account *additional* transformation for the noncommutativity parameter as  $\theta \rightarrow -\theta$ . Hence, NC-SQED, similar to NC-QED, is a **CP**-violating model, but **CPT** invariant.

Next, based on the gauge-fixed Lagrangian (2.1) one can integrated over the auxiliary field  $B$ , so we can derive all the necessary Feynman rules for the propagators and vertex functions

- Gauge field propagator (at Feynman gauge  $\xi = 1$ )

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2}. \quad (2.6)$$

- Scalar field propagator

$$S(k) = \frac{i}{k^2 - m^2}. \quad (2.7)$$

- Ghost field propagator

$$D(k) = \frac{i}{k^2}. \quad (2.8)$$

- Three-point vertex  $\langle \phi^\dagger \phi A \rangle$

$$\Gamma^\mu(p, p') = -ie(p + p')^\mu e^{\frac{i}{2}p \wedge p'}. \quad (2.9)$$

- Four-point vertex  $\langle \phi^\dagger \phi AA \rangle$

$$\Upsilon^{\mu\nu}(p, p', k, k') = 2ie^2 g^{\mu\nu} e^{\frac{i}{2}p \wedge p'} \cos\left(\frac{k \wedge k'}{2}\right). \quad (2.10)$$

- Three-point vertex  $\langle \bar{c} c A \rangle$

$$\Psi^\mu(p, p') = 2ep^\mu \sin\left(\frac{p \wedge p'}{2}\right). \quad (2.11)$$

- The cubic gauge vertex

$$\Omega^{\mu\nu\rho}(p_1, p_2, p_3) = 2e \sin\left(\frac{p_1 \wedge p_2}{2}\right) \left[ g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(p_2 - p_3)^\mu + g^{\rho\mu}(p_3 - p_1)^\nu \right]. \quad (2.12)$$

- The quartic gauge vertex

$$\begin{aligned} \Delta^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = 4ie^2 & \left[ (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \sin\left(\frac{p_1 \wedge p_2}{2}\right) \sin\left(\frac{p_3 \wedge p_4}{2}\right) \right. \\ & + (g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}) \sin\left(\frac{p_3 \wedge p_1}{2}\right) \sin\left(\frac{p_2 \wedge p_4}{2}\right) \\ & \left. + (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) \sin\left(\frac{p_1 \wedge p_4}{2}\right) \sin\left(\frac{p_2 \wedge p_3}{2}\right) \right]. \end{aligned} \quad (2.13)$$

Before proceeding in computing the one-loop corrections to the basic functions we shall now establish the renormalization of the given theory.

### 3 Renormalization

We present the multiplicative renormalization of the NC-SQED. The bare and renormalized fields are related by means of

$$A_\mu^{(0)} = \sqrt{Z_3} A_\mu, \quad \phi^{(0)} = \sqrt{Z_2} \phi, \quad c^{(0)} = \sqrt{\tilde{Z}_3} c, \quad (3.1)$$

Now we rewrite the Lagrangian (2.1) in terms of the renormalized fields, so that we find explicitly for the interaction part

$$\begin{aligned} \mathcal{L}_{\text{int}}^{(r)} = & ieZ_1 \left( \partial^\mu \phi^\dagger \star A_\mu \star \phi - \phi^\dagger \star A_\mu \star \partial^\mu \phi \right) + e^2 Z_4 \phi^\dagger \star A^\mu \star A_\mu \star \phi \\ & - ieZ_{3A} \partial_\mu A_\nu \star [A^\mu, A^\nu]_\star + \frac{e^2}{4} Z_{4A} [A_\mu, A_\nu]_\star \star [A^\mu, A^\nu]_\star + ie\tilde{Z}_1 \partial^\mu \bar{c} \star [A_\mu, c]_\star, \end{aligned} \quad (3.2)$$

we can then introduce the counter-terms as usual  $Z_i = 1 + \delta_i$ . Notice that the renormalization constant  $Z_1$  is related to the vertex  $\langle \phi^\dagger \phi A \rangle$ ,  $Z_{3A}$  is related to the vertex  $\langle AAA \rangle$ ,  $\tilde{Z}_1$  is related to the vertex  $\langle \bar{c}cA \rangle$ ,  $Z_4$  is related to the vertex  $\langle \phi^\dagger \phi AA \rangle$ , and  $Z_{4A}$  is related to the vertex  $\langle AAAA \rangle$ , respectively.

Moreover, the gauge invariance, expressed in terms of the Slavnov-Taylor identities, assures the universality of the (gauge) coupling by renormalization, provided the following identities hold:

$$\frac{Z_{4A}}{Z_{3A}} = \frac{Z_{3A}}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_1}{Z_2} = \frac{Z_4}{Z_1}. \quad (3.3)$$

Since we are interested in computing the basic functions  $\langle \phi^\dagger \phi \rangle$ ,  $\langle AA \rangle$ ,  $\langle \bar{c}c \rangle$ ,  $\langle \phi^\dagger \phi A \rangle$  and  $\langle \phi^\dagger \phi AA \rangle$ , we shall find by this analysis the respective renormalization constants:  $Z_2$ ,  $Z_3$ ,  $\tilde{Z}_3$ ,  $Z_1$  and  $Z_4$ , so that in this case the renormalized coupling constant can be determined as  $e_0 = Z_e e$ , where we have defined conveniently  $Z_e = Z_1 Z_3^{-1/2} Z_2^{-1}$  or  $Z_e = Z_4^{1/2} Z_3^{-1/2} Z_2^{-1/2}$ , and further relations for the renormalization constant  $Z_e$  can be obtained by making use of the relations (3.3). We note that, however, by determining these 5 renormalization constants the remaining 3 ones are immediately determined by means of the gauge identities (3.3). However, the one-loop analysis of the quantum corrections to the cubic and quartic gauge vertices, which yields  $Z_{3A}$  and  $Z_{4A}$ , has already been presented in [18].

To conclude, one can work out the renormalization group equation for the renormalized Green's function, establishing the invariance of the observables under changes of the renormalization scale  $\mu$ . In particular, let us consider the two-point function for the gauge field

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(e) \frac{\partial}{\partial e} - 2\gamma_A(e) \right] \Gamma_{\text{ren}}^{\mu\nu}(p, e, \mu) = 0, \quad (3.4)$$

where we have defined the  $\beta$  and  $\gamma$  functions as follows

$$\beta(e) = \lim_{\epsilon \rightarrow 0} \mu \frac{de}{d\mu}, \quad \gamma_A(e) = \lim_{\epsilon \rightarrow 0} \frac{\mu}{2} \frac{d}{d\mu} \ln Z_3. \quad (3.5)$$

Moreover, if we can consider the other two two-point functions, we find the anomalous dimensions for the scalar and ghost fields,  $\gamma_\phi = \frac{\mu}{2} \frac{d}{d\mu} \ln Z_2$  and  $\gamma_c = \frac{\mu}{2} \frac{d}{d\mu} \ln \tilde{Z}_3$ , respectively. We now proceed in computing the one-loop order radiative corrections to the basic functions, so that the respective renormalization constants can be determined, allowing us to find the basic  $\beta$  and  $\gamma$  functions.

## 4 Radiative Corrections

In this section we shall compute the one-loop correction to the scalar self-energy, polarization tensor, ghost self-energy, and to the three-point and four-point vertex parts  $\langle \phi^\dagger \phi A \rangle$  and  $\langle \phi^\dagger \phi AA \rangle$ . With these expressions we shall then proceed to determine the  $\beta$ -function for the NC-SQED.

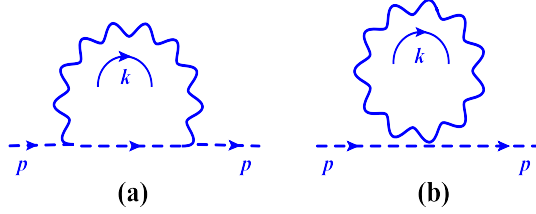


Figure 1: One-loop self-energy graphs for the charged scalar field.

#### 4.1 One-loop scalar self-energy

We start by computing now the simplest one-loop contribution that is the scalar self-energy. The corresponding graphs are depicted at Fig. 1. Their explicit expressions read

$$\begin{aligned}\Sigma_{(a)}(p) &= -e^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{(2p-k)^2}{k^2 ((p-k)^2 - m^2)}, \\ \Sigma_{(b)}(p) &= 2de^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2},\end{aligned}\tag{4.1}$$

that the momentum integral on the contribution (b) is vanishing by dimensional regularization, i.e.  $\Sigma_b(p) = 0$ . Moreover, we see that no noncommutative phase factor is present at contribution (a), consequently this self-energy function is free of noncommutativity effects at least at one-loop order. The remaining momentum integral can be computed by standard Feynman integration and dimensional regularization, and it results into

$$\Sigma_{1-loop}(p) = \Sigma_{(a)}(p) + \Sigma_{(b)}(p) = \Sigma_{(a)}(p) = -\frac{ie^2}{4\pi^2\epsilon} \left( p^2 + \frac{m^2}{2} \right) + \text{finite},\tag{4.2}$$

where we have defined by simplicity  $\frac{2}{\epsilon'} = \frac{2}{\epsilon} - \gamma + \log(4\pi\mu^2)$ , and  $\epsilon = 4-d \rightarrow 0^+$ . Now, making use of the renormalized Lagrangian, we find that the renormalization conditions imply that

$$-\frac{ie^2}{4\pi^2\epsilon} \left( p^2 + \frac{m^2}{2} \right) + i(\delta_2 p^2 - \delta_m) = 0,\tag{4.3}$$

so the expressions for the counter-terms are

$$\delta_2 = \frac{e^2}{4\pi^2\epsilon'}, \quad \delta_m = -\frac{e^2 m^2}{8\pi^2\epsilon'}.\tag{4.4}$$

In particular, we can then make use of  $Z_2 = 1 + \delta_2$  and find the renormalization constant of the matter sector as follows

$$Z_2 = 1 + \frac{e^2}{4\pi^2\epsilon'},\tag{4.5}$$

which is the same as the commutative case.

#### 4.2 One-loop photon self-energy

We have five graphs contributing to the photon polarization tensor at one-loop order, these are given in Fig. 2. The interaction among the gauge and charged scalar fields are given by graphs (a) and (b), graph (c) comes from the cubic self-coupling of the gauge-field, graph (d)



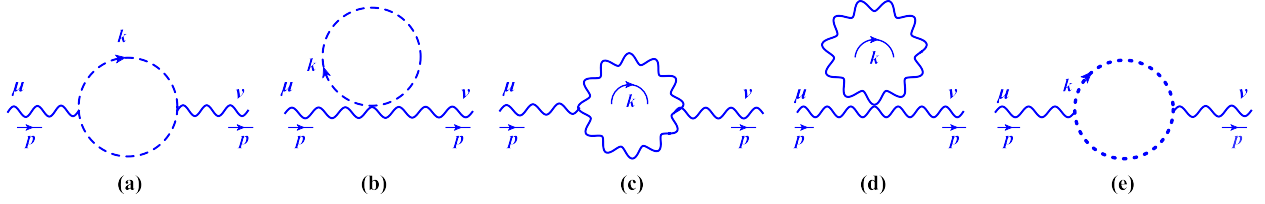


Figure 2: One-loop self-energy graphs for the gauge field.

is the tadpole contribution and graph (e) comes from the ghost loop. Let us consider first the contributions from graphs (a) and (b)

$$\begin{aligned}\Pi_{(a)}^{\mu\nu}(p) &= e^2 \mu^{4-d} N_B \int \frac{d^d k}{(2\pi)^d} \frac{(2k-p)^\mu (2k-p)^\nu}{(k^2 - m^2)((k-p)^2 - m^2)}, \\ \Pi_{(b)}^{\mu\nu}(p) &= -2e^2 \mu^{4-d} N_B \int \frac{d^d k}{(2\pi)^d} \frac{g^{\mu\nu}}{k^2 - m^2}.\end{aligned}\quad (4.6)$$

Notice once again the absence of noncommutative effects in these contributions. It is convenient to rewrite these two contributions together, so that using the Feynman parametrization method we have

$$\begin{aligned}\Pi_{a+b}^{\mu\nu}(p) &= e^2 \mu^{4-d} N_B \int_0^1 dy \int \frac{d^d Q}{(2\pi)^d} \left[ \frac{(1 + 4y^2 - 4y)p^\mu p^\nu + (-4y^2 + 6y - 2)g^{\mu\nu} p^2}{(Q^2 - \Delta)^2} \right. \\ &\quad \left. - \frac{2g^{\mu\nu}}{(Q^2 - \Delta)} + \frac{\frac{4}{d}g^{\mu\nu} Q^2}{(Q^2 - \Delta)^2} \right],\end{aligned}\quad (4.7)$$

where  $Q = k - yp$  and  $\Delta = y(y-1)p^2 + m^2$ . The remaining integration can be readily evaluated and we find

$$\Pi_{a+b}^{\mu\nu}(p) = \frac{ie^2}{24\pi^2 \epsilon'} (p^\mu p^\nu - g^{\mu\nu} p^2) N_B + \text{finite}, \quad (4.8)$$

where  $N_B$  is the number of independent scalar bosons with charge  $\pm 1$ . As one should naively expect from the results from ordinary scalar QED, the above result is consistent with Ward identity. The remaining three diagrams (c), (d) and (e) are given by

$$\Pi_{(c)}^{\mu\nu}(p) = e^2 \mu^{4-d} C_{(c)} \int \frac{d^d k}{(2\pi)^d} \left( \frac{1 - \cos(k \wedge p)}{k^2(p+k)^2} \right) N_{(c)}^{\mu\nu}, \quad (4.9)$$

$$\Pi_{(d)}^{\mu\nu}(p) = e^2 \mu^{4-d} C_{(d)} \int \frac{d^d k}{(2\pi)^d} \left( \frac{1 - \cos(k \wedge p)}{k^2(p+k)^2} \right) N_{(d)}^{\mu\nu}, \quad (4.10)$$

$$\Pi_{(e)}^{\mu\nu}(p) = e^2 \mu^{4-d} C_{(e)} \int \frac{d^d k}{(2\pi)^d} \left( \frac{1 - \cos(k \wedge p)}{k^2(p+k)^2} \right) N_{(e)}^{\mu\nu}, \quad (4.11)$$

where their tensor structures at numerator are

$$\begin{aligned}N_{(c)}^{\mu\nu} &= 2 \left[ (d-6)p^\mu p^\nu + (2d-3)(p^\mu k^\nu + k^\mu p^\nu) + (4d-6)k^\mu k^\nu + (5p^2 + 2k^2 + 2p \cdot k)g^{\mu\nu} \right], \\ N_{(d)}^{\mu\nu} &= 4g^{\mu\nu}(p+k)^2(d-1), \\ N_{(e)}^{\mu\nu} &= 2(p+k)^\nu k^\mu,\end{aligned}\quad (4.12)$$

and also notice that the symmetry factors of them are  $C_{(c)} = \frac{1}{2}$ ,  $C_{(d)} = \frac{1}{2}$  and  $C_{(e)} = -1$ , respectively. Moreover, we can cast all these three contributions into a common expression

$$\Pi_{c+d+e}^{\mu\nu} = \Pi_{(c)}^{\mu\nu} + \Pi_{(d)}^{\mu\nu} + \Pi_{(e)}^{\mu\nu}, \quad (4.13)$$

so that it reads

$$\Pi_{c+d+e}^{\mu\nu} = e^2 \mu^{4-d} \int_0^1 dx \int \frac{d^d Q}{(2\pi)^d} \left( \frac{1 - \cos[(Q - xp) \wedge p]}{(Q^2 - \Delta_1)^2} \right) N_{c+d+e}^{\mu\nu}, \quad (4.14)$$

the tensor structure is given as follows

$$\begin{aligned} N_{c+d+e}^{\mu\nu} = & \left( d - 6 + x^2(4d - 8) - x(4d - 8) \right) p^\mu p^\nu + \left( 3 + 2d + 2dx^2 - x(4d - 2) \right) p^2 g^{\mu\nu}, \\ & + (4d - 8) Q^\mu Q^\nu + 2d Q^2 g^{\mu\nu}, \end{aligned} \quad (4.15)$$

where  $Q = k + xp$  and  $\Delta_1 = x(x - 1)p^2$ .

From Eq. (4.14) we find for the first time the presence of noncommutative effects on the one-loop functions, so that noncommutativity will engender new features on the  $\beta$ -function expression as we expected. We shall next compute separately the planar and non-planar contributions. The planar part is evaluated straightforwardly and it reads

$$(\Pi_{c+d+e}^{\mu\nu})_p = \frac{ie^2}{8\pi^2\epsilon'} \frac{10}{3} \left( p^2 g^{\mu\nu} - p^\mu p^\nu \right) + \text{finite}. \quad (4.16)$$

The non-planar contribution of (4.14) can be computed with help of the following integrals

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - s^2)^\alpha} e^{ik \wedge q} = \frac{2i(-)^\alpha}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(\alpha)} \frac{1}{(s^2)^{\alpha - \frac{d}{2}}} \left( \frac{|\tilde{k}|s}{2} \right)^{\alpha - \frac{d}{2}} K_{\alpha - \frac{d}{2}}(|\tilde{k}|s), \quad (4.17)$$

and

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2 - s^2)^\alpha} e^{ik \wedge q} = \eta^{\mu\nu} F_\alpha + \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^2} G_\alpha, \quad (4.18)$$

where

$$\{F_\alpha, G_\alpha\} = \frac{i(-)^{\alpha-1}}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(\alpha)} \frac{1}{(s^2)^{\alpha-1-\frac{d}{2}}} \{f_\alpha, g_\alpha\}, \quad (4.19)$$

with the definitions

$$f_\alpha = \left( \frac{s|\tilde{k}|}{2} \right)^{\alpha-1-\frac{d}{2}} K_{\alpha-1-\frac{d}{2}}(|\tilde{k}|s), \quad (4.20)$$

$$g_\alpha = (2\alpha - 2 - d) \left( \frac{s|\tilde{k}|}{2} \right)^{\alpha-1-\frac{d}{2}} K_{\alpha-1-\frac{d}{2}}(|\tilde{k}|s) - 2 \left( \frac{s|\tilde{k}|}{2} \right)^{\alpha-\frac{d}{2}} K_{\alpha-\frac{d}{2}}(|\tilde{k}|s). \quad (4.21)$$

Nonetheless, we find that the non-planar part is finite when  $\epsilon = 4 - \omega \rightarrow 0^+$  and does not have any effect on the  $\beta$ -function. It is notable that although the non-planar part does not

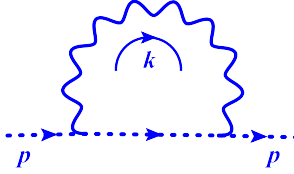


Figure 3: One-loop self-energy graph for the ghost field.

contribute to the  $\beta$ -function, the planar part (4.16) has origin precisely from the noncommutativity, since graphs (c), (d) and (e) do not appear in the ordinary theory. In this way noncommutativity effects are encoded in the contribution (4.16).

Hence, the complete one-loop order correction to the photon self-energy is given by

$$\Pi_{1-loop}^{\mu\nu} = \frac{ie^2}{16\pi^2\epsilon'} \left( \frac{20}{3} - \frac{2N_B}{3} \right) \left( p^2 g^{\mu\nu} - p^\mu p^\nu \right) + \text{finite}, \quad (4.22)$$

and by taking into account the renormalized polarization tensor, the renormalization condition shows that the counter-term  $\delta_3$  satisfies the relation

$$\frac{ie^2}{16\pi^2\epsilon'} \left( \frac{20}{3} - \frac{2N_B}{3} \right) \left( p^2 g^{\mu\nu} - p^\mu p^\nu \right) - i\delta_3 (g^{\mu\nu} p^2 - p^\mu p^\nu) = 0. \quad (4.23)$$

Finally, by means of  $Z_3 = 1 + \delta_3$ , we can compute the renormalization constant related to the photon sector

$$Z_3 = 1 + \frac{e^2}{16\pi^2\epsilon'} \left( \frac{20}{3} - \frac{2N_B}{3} \right). \quad (4.24)$$

This is one of the main results in order to determine the  $\beta$ -function. However, one might wish to compare (4.24) with the expression from the commutative scalar QED, where the renormalization constant of the gauge field comes solely from (4.8) and is given by  $Z_3 = 1 - \frac{e^2 N_B}{24\pi^2\epsilon'}$ . It is worth mentioning that in commutative Yang-Mills gauge theories coupled to the scalar fields with an arbitrary number of bosons, the renormalization constant of the non-abelian gauge field is described by [19]

$$Z_3 = 1 + \frac{g^2}{16\pi^2\epsilon'} \left( \frac{10}{3} C_2(G) - \frac{2N_B}{3} C(r) \right), \quad (4.25)$$

where  $C_2(G)$  is the quadratic Casimir operator of the adjoint representation and  $C(r)$  is a constant for each representation.

### 4.3 One-loop ghost self-energy

The calculation of the ghost self-energy is straightforwardly since it comes from only one graph, depicted in Fig. 3. The given contribution reads

$$G(p) = 2e^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{p \cdot (p-k)}{k^2 (p-k)^2} (1 - e^{ik \wedge p}). \quad (4.26)$$

The planar part of this expression yields

$$(G)_p(p) = \frac{ie^2}{8\pi^2\epsilon'} p^2 + \text{finite}, \quad (4.27)$$

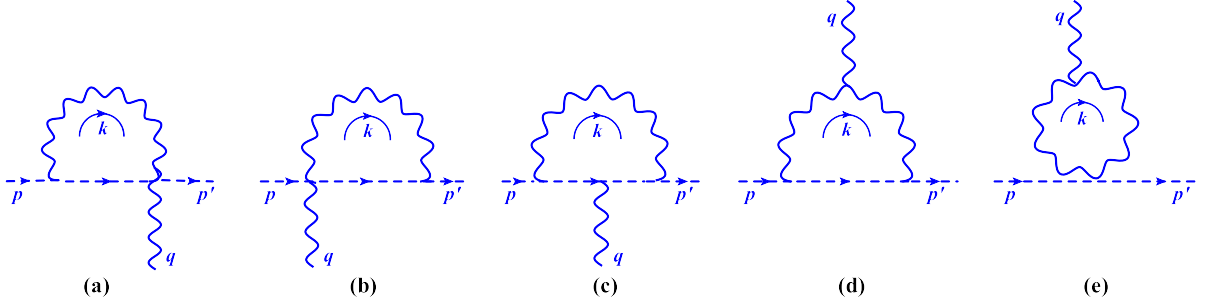


Figure 4: One-loop graphs contributing to three-point vertex  $\langle \phi^\dagger \phi A \rangle$ .

while the non-planar part is, once again, finite. Thus, the respective ghost renormalization constant by making use of  $\tilde{Z}_3 = 1 + \delta_3$  is given by

$$\tilde{Z}_3 = 1 + \frac{e^2}{8\pi^2\epsilon'}, \quad (4.28)$$

and it plays an important part in establishing the full set of renormalization constants.

#### 4.4 One-loop correction to the three-point vertex part $\langle \phi^\dagger \phi A \rangle$

We now turn our attention to the calculation of the one-loop correction to the vertex part  $\langle \phi^\dagger \phi A \rangle$ . Here  $p$  and  $p'$  are the momenta of the incident and emergent scalar fields, and  $q = p - p'$  is the transferred momentum to the gauge field. The corresponding diagrams are shown at Fig. 4.

Let us start by computing the contributions from graphs (a) and (b)

$$\Lambda_{(a)}^\mu(p, p') = e^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{(2p - k)^\mu}{k^2 ((p - k)^2 - m^2)} e^{\frac{i}{2} p \wedge p'} (e^{ik \wedge q} + 1), \quad (4.29)$$

$$\Lambda_{(b)}^\mu(p, p') = e^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{(2p' - k)^\mu}{k^2 ((p' - k)^2 - m^2)} e^{\frac{i}{2} p \wedge p'} (e^{ik \wedge q} + 1). \quad (4.30)$$

We observe that apart from the phase factor  $e^{\frac{i}{2} p \wedge p'}$  diagrams (a) and (b) are related by the symmetry replacement  $p \leftrightarrow p'$ . In this way we can compute the contribution of (b) based on the results of (a). We proceed now to compute separately the planar part of (4.29)

$$\begin{aligned} \Lambda_{(a)}^\mu|_p &= e^2 \mu^{4-d} e^{\frac{i}{2} p \wedge p'} \int \frac{d^d k}{(2\pi)^d} \frac{(2p - k)^\mu}{k^2 ((p - k)^2 - m^2)} \\ &= \frac{3ie^2}{16\pi^2\epsilon'} p^\mu e^{\frac{i}{2} p \wedge p'} + \text{finite}, \end{aligned} \quad (4.31)$$

and with help of (4.17) we can compute the non-planar part

$$\Lambda_{(a)}^\mu|_{n-p} = e^2 \mu^{4-d} e^{\frac{i}{2} p \wedge p'} \int \frac{d^d k}{(2\pi)^d} \frac{(2p - k)^\mu}{k^2 ((p - k)^2 - m^2)} e^{ik \wedge q}, \quad (4.32)$$

and show that it is also finite when  $\epsilon = 4 - d \rightarrow 0^+$ . Based on the above discussion, on the symmetry argument, we can determine the contribution of graph (b) from (4.31) as being

$$\Lambda_{(b)}^\mu|_p = \frac{3ie^2}{16\pi^2\epsilon'} p'^\mu e^{\frac{i}{2} p \wedge p'} + \text{finite}, \quad (4.33)$$

$$\Lambda_{(b)}^\mu|_{n-p} = \text{finite}. \quad (4.34)$$

Finally, we find that the contributions of the diagrams (a) and (b) can be expressed conveniently as

$$\Lambda_{(a+b)}^\mu|_p = \frac{3ie^2}{16\pi^2\epsilon'}(p+p')^\mu e^{\frac{i}{2}p\wedge p'} + \text{finite}. \quad (4.35)$$

Moreover, the expression of the graph (c) is given by

$$\Lambda_{(c)}^\mu(p, p') = -e^2\mu^{4-d}e^{\frac{i}{2}p\wedge p'} \int \frac{d^d k}{(2\pi)^d} \frac{(2p-k)\cdot(2p'-k)(p+p'-2k)^\mu}{k^2((p'-k)^2-m^2)((p-k)^2-m^2)} e^{ik\wedge q}, \quad (4.36)$$

and we can compute with help of (4.17) and (4.18). But first notice that this graph is purely non-planar and finite when the limit of  $\epsilon = 4-d \rightarrow 0^+$  is taken. Hence, this diagram does not contribute to the  $\beta$ -function

$$\Lambda_{(c)}^\mu(p, p') = \text{finite}, \quad (4.37)$$

while its commutative counterpart is divergent in this limit. So far we have computed diagrams similar to those of ordinary scalar QED. The following contributions, graphs (d) and (e), are engendered by noncommutativity, so we shall actually obtain the noncommutative contribution to this vertex function. We compute next the diagram (d), whose expression is as follows

$$\Lambda_{(d)}^\mu = e^2\mu^{4-d}e^{\frac{i}{2}(p\wedge p')}\Gamma(3) \int_0^1 dy \int_0^{1-y} dz \int \frac{d^d Q}{(2\pi)^d} \frac{(1 - e^{-iq\wedge(Q+zp)})}{(Q^2 - \Delta)^3} N_{(d)}^\mu, \quad (4.38)$$

where  $\Delta = (yq + zp)^2 - yq^2 - zp^2 + zm^2$ ,  $Q = k - yq - zp$  and the numerator  $N_{(d)}^\mu$  is defined as

$$\begin{aligned} N_{(d)}^\mu = & [Q - (1+y)p' + (1+y+z)p] \cdot [(1-y-z)p + (1+y)p' - Q] [(2-z-y)p + yp' - Q]^\mu \\ & + [(2-z-y)p + yp' - Q] \cdot [Q + (2-y)p' + (z+y-2)p] [(1-y-z)p + (1+y)p' - Q]^\mu \\ & + [(2-z-y)p - Q + yp'] \cdot [(1-y-z)p + (1+y)p' - Q] [(1-2y-2z)p - (1-2y)p' - 2Q]^\mu. \end{aligned} \quad (4.39)$$

We can proceed and separate the different powers of  $Q$  in the numerator, so that after computing the momentum and Feynman integrals we obtain the following result for the planar part

$$\Lambda_{(d)}^\mu|_p = -\frac{3ie^2}{16\pi^2\epsilon'}(p+p')^\mu e^{\frac{i}{2}p\wedge p'} + \text{finite}, \quad (4.40)$$

while, we see that the non-planar contribution is finite in this case  $\Lambda_{(d)}^\mu|_{n-p} = \text{finite}$ . Hence, the contribution of this diagram to the respective renormalization constant is precisely

$$\Lambda_{(d)}^\mu = -\frac{3ie^2}{16\pi^2\epsilon'}(p+p')^\mu e^{\frac{i}{2}p\wedge p'} + \text{finite}. \quad (4.41)$$

This is the first noncommutative contribution to this vertex function (also to the respective renormalization constant), we shall next compute the diagram (e), which is also due to noncommutativity. The graph (e) has the following expression

$$\Lambda_{(e)}^\mu = -2e^2\mu^{4-d}(1-d)e^{\frac{i}{2}p\wedge p'} \int_0^1 dy \int \frac{d^\omega Q}{(2\pi)^d} \frac{2Q^\mu e^{iq\wedge Q}}{(Q^2 - \Delta)^2}, \quad (4.42)$$

with  $Q = k - yq$  and  $\Delta = y(y-1)q^2$ . Once again we see a purely non-planar graph, so when the limit  $\epsilon = 4 - d \rightarrow 0^+$  is taken we get a finite result

$$\Lambda_{(e)}^\mu = \text{finite}. \quad (4.43)$$

Therefore, the total one-loop contribution is found by summing all the obtained results Eqs. (4.35), (4.37), (4.41) and (4.43), thus

$$\Lambda_{1-loop}^\mu(p) = \text{finite}. \quad (4.44)$$

We note that the divergent parts of (4.35) and (4.41) are mutually cancelled, so that the one-loop order correction to the vertex part  $\Lambda_{1-loop}^\mu(p)$ , corresponding to the vertex  $\langle \phi^\dagger \phi A \rangle$ , is finite due to noncommutative effects. This result is in contrast to its commutative counterpart which gives a divergent contribution to the vertex part. In fact, since we have a finite expression for the one-loop correction, the renormalization condition for the vertex part yields

$$\Lambda_{1-loop}^\mu(p) \Big|_{div.} - i\delta_1(p+p')^\mu e^{\frac{i}{2}p \wedge p'} = 0, \quad (4.45)$$

thus we conclude that  $\delta_1 = 0$ , and consequently the renormalization constant reads

$$Z_1 = 1. \quad (4.46)$$

Surprisingly we see that the noncommutativity contribution to the vertex part  $\Lambda_{1-loop}^\mu(p)$  is such that the ordinary divergent terms are exactly cancelled, rendered a finite expression at one-loop order, unlike the commutative result which reads  $Z_1|_{com.} = 1 + \frac{e^2}{4\pi^2\epsilon}$ .

#### 4.5 One-loop correction to the four-point vertex part $\langle \phi^\dagger \phi AA \rangle$

The last point that we will discuss in this section, in order to compute  $\beta$  function, is the calculation of the four-point vertex part  $\langle \phi^\dagger \phi AA \rangle$ , which is the fourth basic function and it is related to Compton's scattering. The twelve diagrams that contribute to this function at one-loop order are shown in the Fig. 5. We realize that the diagrams (a), (c), (f), (g) and (j) are the same from the ordinary commutative theory, while the diagrams (b), (d), (e), (h), (i), (k) and (l) are originated from the noncommutativity. For this reason we will discuss these two classes of diagrams separately in order to highlight the part played by the noncommutative effects.

Let us summarize the expressions corresponding to the (commutative) diagrams (a), (c), (f), (g) and (j) written in a simplified form

$$\Xi_{(a)}^{\mu\nu} = -e^2 \mu^{4-d} g^{\mu\nu} e^{\frac{i}{2}p \wedge p'} e^{\frac{i}{2}s \wedge q} \int \frac{d^d k}{(2\pi)^\omega} \frac{[1 + e^{ik \wedge q} + e^{ik \wedge (q-s)} + e^{-ik \wedge s}]}{k^2 ((p' - s - k)^2 - m^2)}, \quad (4.47)$$

$$\begin{aligned} \Xi_{(c)}^{\mu\nu} &= e^2 \mu^{4-d} g^{\mu\nu} e^{-\frac{i}{2}p' \wedge p} \left( e^{\frac{i}{2}s \wedge q} + e^{-\frac{i}{2}s \wedge q} \right) \\ &\times \int \frac{d^d k}{(2\pi)^d} \frac{(k + q - p' - s) \cdot (k - p)}{(p + k)^2 (k^2 - m^2) ((s - q - k)^2 - m^2)} e^{\frac{i}{2}k \wedge (p' - 2p)}, \end{aligned} \quad (4.48)$$

$$\begin{aligned} \Xi_{(f)}^{\mu\nu} &= e^2 \mu^{4-d} e^{\frac{i}{2}q \wedge p'} e^{\frac{i}{2}p \wedge s} \\ &\times \int \frac{d^d k}{(2\pi)^d} \frac{(2p - k)^\mu (2p - 2k + s)^\nu}{k^2 ((p - k + s)^2 - m^2) ((p - k)^2 - m^2)} [e^{i(s-q) \wedge k} + e^{is \wedge k}], \end{aligned} \quad (4.49)$$

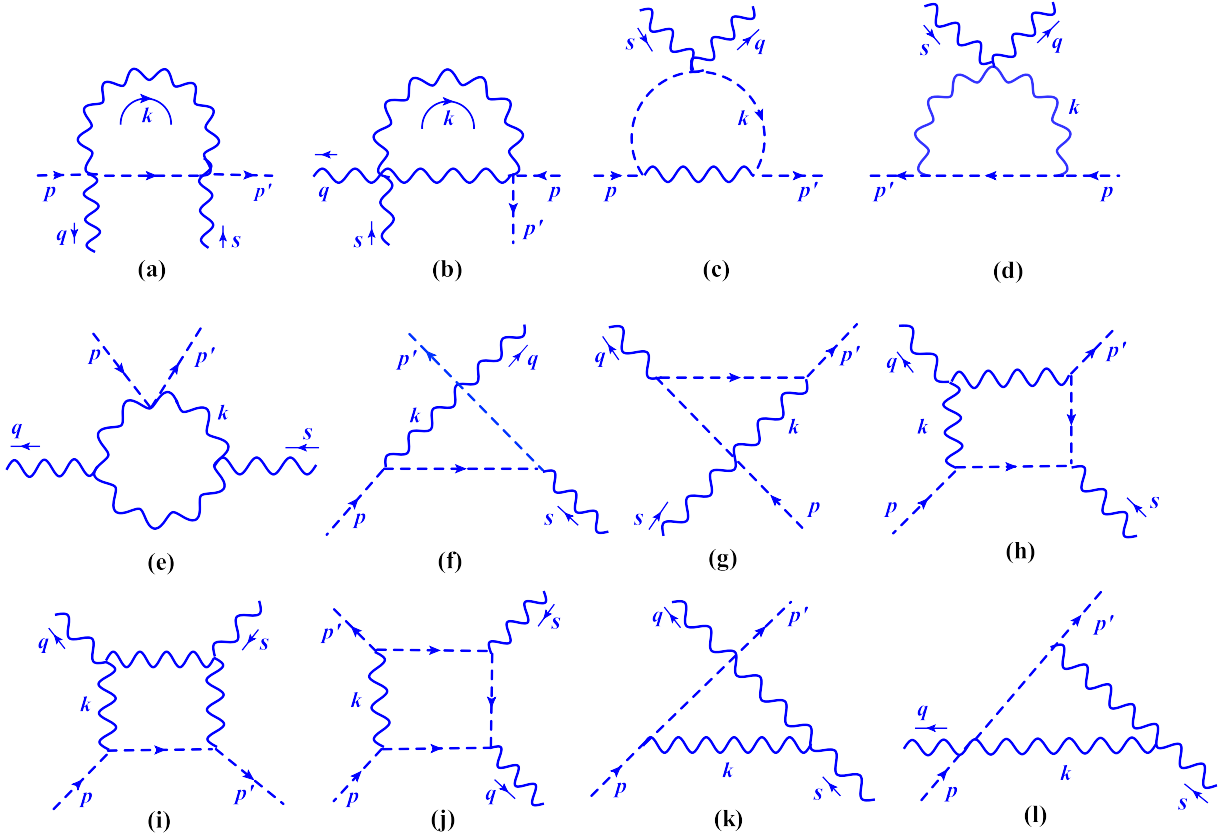


Figure 5: One-loop graphs contributing to four-point vertex  $\langle \phi^\dagger \phi A A \rangle$ .

$$\begin{aligned} \Xi_{(g)}^{\mu\nu} &= e^2 \mu^{4-d} e^{\frac{i}{2} q \wedge p'} e^{\frac{i}{2} p \wedge s} \\ &\times \int \frac{d^d k}{(2\pi)^d} \frac{(2p' - k)^\mu (p' + p - 2k + s)^\nu}{k^2 ((p - k + s)^2 - m^2) ((p' - k)^2 - m^2)} [e^{i(s-q) \wedge k} + e^{iq \wedge k}], \end{aligned} \quad (4.50)$$

$$\begin{aligned} \Xi_{(j)}^{\mu\nu} &= -e^2 \mu^{4-d} e^{\frac{i}{2} p \wedge s} e^{\frac{i}{2} q \wedge p'} \\ &\times \int \frac{d^d k}{(2\pi)^d} \frac{(2p' - 2k + q)^\mu (2p - 2k + s)^\nu (2p - k) \cdot (2p' - k)}{k^2 ((p - k)^2 - m^2) ((p - k + s)^2 - m^2) ((p' - k)^2 - m^2)} e^{i(s-q) \wedge k}. \end{aligned} \quad (4.51)$$

By a simple analysis we readily conclude that the contributions from diagrams (c), (f), (g) and (j), Eqs. (4.48), (4.49), (4.50) and (4.51) are non-planar, which actually means that they are finite when  $\epsilon = 4 - d \rightarrow 0^+$ , showing that they do not contribute to the renormalization constant  $Z_4$  (i.e. to the  $\beta$ -function). On the other hand, for the graph (a) Eq. (4.47) we find a contribution from the planar part to the  $\beta$ -function, so that the full expression reads

$$\Xi_{(a)}^{\mu\nu}(p) = -\frac{ie^2}{4\pi^2 \epsilon'} g^{\mu\nu} e^{\frac{i}{2} p \wedge p'} \cos\left(\frac{q \wedge s}{2}\right) + \text{finite}. \quad (4.52)$$

The remaining contributions, which consist of the second class of diagrams (b), (e), (i),

(k) and (l), coming from the noncommutativity, are given as follows

$$\Xi_{(b)}^{\mu\nu} = \frac{3ie^2}{4\pi^2\epsilon'} g^{\mu\nu} e^{\frac{i}{2}p\wedge p'} \cos\left(\frac{q\wedge s}{2}\right) + \text{finite}, \quad (4.53)$$

$$\Xi_{(e)}^{\mu\nu} = \frac{9ie^2}{16\pi^2\epsilon'} g^{\mu\nu} e^{\frac{i}{2}p\wedge p'} \cos\left(\frac{q\wedge s}{2}\right) + \text{finite}, \quad (4.54)$$

$$\Xi_{(i)}^{\mu\nu} = \frac{-3ie^2}{16\pi^2\epsilon'} g^{\mu\nu} e^{\frac{i}{2}p\wedge p'} \cos\left(\frac{q\wedge s}{2}\right) + \text{finite}, \quad (4.55)$$

$$\Xi_{(k)}^{\mu\rho} = \frac{-3ie^2}{16\pi^2\epsilon'} g^{\mu\nu} e^{\frac{i}{2}p\wedge p'} \cos\left(\frac{q\wedge s}{2}\right) + \text{finite}, \quad (4.56)$$

$$\Xi_{(\ell)}^{\mu\nu} = \frac{-3ie^2}{16\pi^2\epsilon'} g^{\mu\nu} e^{\frac{i}{2}p\wedge p'} \cos\left(\frac{q\wedge s}{2}\right) + \text{finite}, \quad (4.57)$$

while the results from graphs (d) and (h) are purely non-planar, and thus give simply a finite contribution

$$\Xi_{(d)}^{\mu\nu} = \text{finite}, \quad (4.58)$$

$$\Xi_{(h)}^{\mu\nu} = \text{finite}. \quad (4.59)$$

We can finally determine the complete one-loop order correction to the four-point vertex function by summing all the twelve contributions computed above, so that it yields

$$\Xi_{1-loop}^{\mu\nu} = \frac{ie^2}{2\pi^2\epsilon'} g^{\mu\nu} e^{\frac{i}{2}p\wedge p'} \cos\left(\frac{s\wedge q}{2}\right) + \text{finite}. \quad (4.60)$$

The respective renormalization condition for the four-point vertex function implies that the divergent part satisfies the relation

$$\Xi_{1-loop}^{\mu\nu} \Big|_{div.} + 2ie^2\delta_4 g^{\mu\nu} e^{\frac{i}{2}p\wedge p'} \cos\left(\frac{s\wedge q}{2}\right) = 0, \quad (4.61)$$

so that the counter-term  $\delta_4$  can readily be evaluated. Hence, the relevant renormalization constant is obtained by  $Z_4 = 1 + \delta_4$ ,

$$Z_4 = 1 - \frac{e^2}{4\pi^2\epsilon'}. \quad (4.62)$$

With this expression we conclude the section regarding the calculation of radiative correction, and next we use the renormalizability analysis previously established in order to compute basic  $\beta$  and  $\gamma$  functions.

## 5 $\beta$ -function and anomalous dimensions

Finally, based on the renormalization analysis developed at Sec. 3, we have that the  $\beta$ -function in this case is given by

$$\beta(e) = \mu \frac{de(\mu)}{d\mu}, \quad (5.1)$$

where we recall that the relation between bare and dressed coupling is given either by  $e_0 = eZ_3^{-1/2}Z_2^{-1}Z_1$  or alternatively by  $eZ_4^{1/2}Z_3^{-1/2}Z_2^{-1/2}$ . Hence, by making use of the results for the



renormalization constants  $Z_2$ ,  $Z_3$ ,  $\tilde{Z}_3$ ,  $Z_1$  and  $Z_4$ , Eqs. (4.5), (4.24), (4.28), (4.46) and (4.62), respectively, we find for the one-loop order  $\beta$ -function

$$\beta(e)\Big|_{\text{NC-SQED}} = -\frac{e^3}{16\pi^2} \left( \frac{22}{3} - \frac{N_B}{3} \right). \quad (5.2)$$

We observe that for small values of  $N_B$ , the sign of the  $\beta$ -function is negative and consequently the theory is asymptotically free, similar to the  $\beta$ -function of the non-abelian gauge theories coupled to the matter with a small number of boson or fermion flavors. Moreover, the anomalous dimensions of the scalar, gauge and ghosts fields are readily obtained

$$\gamma_\phi = \frac{e^2}{8\pi^2}, \quad \gamma_A = \frac{e^2}{16\pi^2} \left( \frac{10}{3} - \frac{N_B}{3} \right), \quad \gamma_c = \frac{e^2}{32\pi^2}. \quad (5.3)$$

The result (5.2) can be compared to the one-loop  $\beta$ -function of NC-QED [8],

$$\beta(e)\Big|_{\text{NC-QED}} = -\frac{e^3}{16\pi^2} \left( \frac{22}{3} - \frac{4N_F}{3} \right). \quad (5.4)$$

We see that both results are similar and just the contribution of the matter sector is different. Indeed in the absence of the matter sector, the contribution of the gauge part to the  $\beta$ -function is the same and this shows the correctness of our result. Besides, the coefficient appearing in the matter part of the NC-QED is four times that of the NC-SQED. This, indeed, comes from the trace over the gamma matrices in  $d = 4$ , that indicates the spinor nature of the matter field in the NC-QED that actually, when compared to the NC-SQED, has intrinsic angular momentum, spin.

We may as well compare the present result with the  $\beta$ -function of the commutative Yang-Mills theory coupled to the scalar fields

$$\beta(e)\Big|_{\text{SYM}} = -\frac{e^3}{16\pi^2} \left( \frac{11}{3}C_2(G) - \frac{2N_B}{3}C(r) \right), \quad (5.5)$$

in which for gauge fields in the adjoint representation  $C_2(G) = N_c$ , while for matter fields in the fundamental (or anti-fundamental) representation  $C(r) = \frac{1}{2}$ . As we see, the comparison of the results (5.2) and (5.5) yields  $C_2(G) = 2$ . Consequently, the obtained result for the  $\beta$ -function of the NC-SQED (5.2) is exactly the same as the  $\beta$ -function for the Yang-Mills theory in the presence of the scalar matter fields with  $SU(2)$  gauge group [19, 20].

## 6 Concluding remarks

In this paper we have considered the scalar QED defined in a noncommutative spacetime, established its renormalizability and computed the one-loop order radiative corrections. In addition, we have showed the BRST Slavnov transformations leaving the gauge-fixed Lagrangian invariant and discussed how the discrete symmetries are changed under the noncommutative setup.

The multiplicative renormalization has been applied to the NC scalar QED, so that the respective counter-term Lagrangian was obtained. Moreover, the Slavnov-Taylor identities

were used in order to show a series of identities relating all the renormalization constants of the theory. These identities are valuable since they allow us to determine the remaining constants from the knowledge of some renormalization constants without further calculation. To conclude the section we wrote the renormalized group equation for the two-point function for the gauge field in order to introduce the  $\beta$ -function for the gauge coupling and the  $\gamma$ -function for the dynamical fields.

After establishing the renormalization of the model, we proceeded in computing the one-loop order radiative corrections to the basic functions. We focused on those simpler contributions that allowed us to compute the full set of renormalization constants: gauge, charged scalar and ghost fields self-energies, three- and four-point vertex functions  $\langle\phi^\dagger\phi A\rangle$  and  $\langle\phi^\dagger\phi AA\rangle$ , respectively. We called special attention to those graphs that were exclusively from noncommutative nature, going to zero at the commutative limit. Only the counter-term  $\delta_1$  related to the vertex  $\langle\phi^\dagger\phi A\rangle$  had a null value (finite contribution only), all the others counter-terms absorbed the respective divergent part of the self-energy and vertex functions. It should be emphasized that, based on the obtained results, the remaining renormalization constants  $Z_{3A}$ ,  $Z_{4A}$  and  $\tilde{Z}_1$  can immediately be determined by means of the gauge identities (3.3).

To conclude we computed the  $\beta$ -function for the gauge coupling and the anomalous dimensions for the dynamical fields  $(A_\mu, \phi, \bar{c}, c)$ . In order to highlight the obtained result for the  $\beta$ -function, we compare it with the NC QED and commutative SYM  $\beta$ -function expressions – as a matter of fact, the gauge sector is strictly the same and differences are only found in the matter sector contribution.

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